

THE GLOBAL EVOLUTION OF STATES OF A CONTINUUM KAWASAKI MODEL WITH REPULSION

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ABSTRACT. An infinite system of point particles performing random jumps in \mathbb{R}^d with repulsion is studied. The states of the system are probability measures on the space of particle's configurations. The result of the paper is the construction of the global in time evolution of states with the help of the corresponding correlation functions. It is proved that for each initial sub-Poissonian state μ_0 , the constructed evolution $\mu_0 \mapsto \mu_t$ preserves this property. That is, μ_t is sub-Poissonian for all $t > 0$.

1. INTRODUCTION

1.1. Posing. In this paper, we continue dealing with the Kawasaki model studied in [2]. The model describes the evolution of an infinite system of point particles placed in \mathbb{R}^d which perform random jumps with repulsion. The phase space of the model is the set Γ of all subsets $\gamma \subset \mathbb{R}^d$ such that the set $\gamma \cap \Lambda$ is finite whenever $\Lambda \subset \mathbb{R}^d$ is compact. This set is equipped with a measurability structure that allows for considering the probability measures on Γ as states of the system. Among them one may distinguish Poissonian states in which the particles are independently distributed over \mathbb{R}^d . In *sub-Poissonian* states, the dependence between the particle's positions is not too strong (see the next subsection). In [2], the evolution $\mu_0 \mapsto \mu_t$ of the system's states was shown to hold in the set of sub-Poissonian states for $t < T$ with some $T < \infty$. The main result of the present study consists in proving the existence of such an evolution for all $t > 0$. This is the first result of this kind for infinite continuum systems of point particles performing jumps with interaction. The case of free jumps was described in [1, 8].

As was shown in [6], for infinite particle systems with birth-and-death dynamics the states remain sub-Poissonian globally in time if the birth of the particles is in a sense controlled by their death. For conservative dynamics in which the particles just change their positions, the interaction may in general change the sub-Poissonian character of the state in finite time (even cause an explosion), e.g., due to an infinite number of simultaneous correlated jumps. Thus, the conceptual outcome of the present study is that this is not the case for the considered model. The important peculiarity of this

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result is that it has been obtained by methods different from those used in [6]. We believe that a combination of these methods with those of [6] can be of great use in studying evolution of systems in which birth-and-death processes are accompanied by random motion, e.g., individual-based models of disease spread.

1.2. Presenting the result. To characterize states of an infinite particle system one employs *observables* – suitable functions $F : \Gamma \rightarrow \mathbb{R}$. Their evolution is described by the Kolmogorov equation

$$\frac{d}{dt}F_t = LF_t, \quad F_t|_{t=0} = F_0, \quad (1.1)$$

where the operator L specifies the model. In our case, it has the following form

$$(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} c(x, y, \gamma) [F(\gamma \setminus x \cup y) - F(\gamma)] dy, \quad (1.2)$$

with c given in (2.14) below. The evolution of states is supposed to be derived from the Fokker-Planck equation

$$\frac{d}{dt}\mu_t = L^*\mu_t, \quad \mu_t|_{t=0} = \mu_0, \quad (1.3)$$

related to that in (1.1) by the duality

$$\int_{\Gamma} F_t(\gamma) \mu_0(d\gamma) = \int_{\Gamma} F_0(\gamma) \mu_t(d\gamma). \quad (1.4)$$

As is usual for models of this kind, the direct meaning of (1.1) or (1.3) can only be given for states of finite systems, cf. [9]. In this case, the Banach space where the Cauchy problem in (1.3) is defined can be the space of signed measures with finite variation.

In this work, we continue following the approach in which the evolution of states is described without the direct use of (1.3), see [2, 4, 6] and the references therein. To explain its essence let us consider the set of all compactly supported continuous functions $\theta : \mathbb{R}^d \rightarrow (-1, 0]$. For a state μ , its *Bogoliubov* functional [5] is defined as

$$B_\mu(\theta) = \int_{\Gamma} \prod_{x \in \gamma} (1 + \theta(x)) \mu(d\gamma), \quad (1.5)$$

with θ running through the mentioned set of functions. For the homogeneous Poisson measure π_\varkappa , $\varkappa > 0$, the functional (1.5) takes the form

$$B_{\pi_\varkappa}(\theta) = \exp \left(\varkappa \int_{\mathbb{R}^d} \theta(x) dx \right).$$

In state π_\varkappa , the particles are independently distributed over \mathbb{R}^d with density \varkappa . The set of *sub-Poissonian* states $\mathcal{P}_{\text{exp}}(\Gamma)$ is then defined as that containing all those states μ for which B_μ can be continued, as a function of θ , to

an exponential type entire function on $L^1(\mathbb{R}^d)$. This exactly means that B_μ can be written in the form

$$B_\mu(\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} k_\mu^{(n)}(x_1, \dots, x_n) \theta(x_1) \cdots \theta(x_n) dx_1 \cdots dx_n, \quad (1.6)$$

where $k_\mu^{(n)}$ is the n -th order correlation function of the state μ . It is a symmetric element of $L^\infty((\mathbb{R}^d)^n)$ for which

$$\|k_\mu^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} \leq C \exp(\vartheta n), \quad n \in \mathbb{N}_0, \quad (1.7)$$

with some $C > 0$ and $\vartheta \in \mathbb{R}$. Note that $k_{\pi_\pi}^{(n)}(x_1, \dots, x_n) = \pi^n$. Note also that (1.6) can be viewed as an analog of the Taylor expansion of the characteristic function of a probability measure. That is why, $k_\mu^{(n)}$ are also called *moment functions*.

Under standard conditions imposed on the jump kernel c , see (2.14) – (2.16), we prove that the correlation functions evolve $k_{\mu_0}^{(n)} \mapsto k_t^{(n)}$ in such a way that each $k_t^{(n)}$, $t > 0$, is the correlation function of a unique sub-Poissonian measure μ_t , see Theorem 3.5. Moreover, assuming that $k_{\mu_0}^{(n)}$ satisfies (1.7), we show that the following holds

$$\forall t > 0 \quad \forall n \in \mathbb{N}_0 \quad 0 \leq k_t^{(n)}(x_1, \dots, x_n) \leq C \exp(n[\vartheta + \alpha t]),$$

where $\alpha > 0$ is a model parameter, see (2.15).

2. PRELIMINARIES AND THE MODEL

Here we briefly present necessary information on the subject – its more detailed description can be found in [2, 4, 5, 6] and in the literature quoted in these works.

2.1. Configuration spaces. Let $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_b(\mathbb{R}^d)$ denote the sets of all Borel and all bounded Borel subsets of \mathbb{R}^d , respectively. The configuration space Γ mentioned above is equipped with the vague topology and thus with the corresponding Borel σ -field $\mathcal{B}(\Gamma)$. For $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, we set

$$\Gamma_\Lambda = \{\gamma \in \Gamma : \gamma \subset \Lambda\}.$$

Clearly $\Gamma_\Lambda \in \mathcal{B}(\Gamma)$, and hence

$$\mathcal{B}(\Gamma_\Lambda) := \{A \cap \Gamma_\Lambda : A \in \mathcal{B}(\Gamma)\}$$

is a sub-field of $\mathcal{B}(\Gamma)$. Let $p_\Lambda : \Gamma \rightarrow \Gamma_\Lambda$ be the projection $p_\Lambda(\gamma) = \gamma_\Lambda = \gamma \cap \Lambda$. It is clearly measurable, and thus the sets

$$p_\Lambda^{-1}(A_\Lambda) := \{\gamma \in \Gamma : p_\Lambda(\gamma) \in A_\Lambda\}, \quad A_\Lambda \in \mathcal{B}(\Gamma_\Lambda)$$

belong to $\mathcal{B}(\Gamma)$ for each Borel Λ . Let $\mathcal{P}(\Gamma)$ denote the set of all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$. For a given $\mu \in \mathcal{P}(\Gamma)$, its projection on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ is defined as

$$\mu^\Lambda(A_\Lambda) = \mu(p_\Lambda^{-1}(A_\Lambda)), \quad A_\Lambda \in \mathcal{B}(\Gamma_\Lambda). \quad (2.1)$$

Let Γ_0 be the set of all finite $\gamma \in \Gamma$. It is an element of $\mathcal{B}(\Gamma)$ as each of $\gamma \in \Gamma_0$ belongs to a certain Γ_Λ , $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. Note that $\Gamma_\Lambda \subset \Gamma_0$ for each such Λ . Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It can be proved that a function $G : \Gamma_0 \rightarrow \mathbb{R}$ is $\mathcal{B}(\Gamma)/\mathcal{B}(\mathbb{R})$ -measurable if and only if, for each $n \in \mathbb{N}_0$, there exists a symmetric Borel function $G^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ such that

$$G(\eta) = G(\eta) = G^{(n)}(x_1, \dots, x_n), \quad (2.2)$$

for $\eta = \{x_1, \dots, x_n\}$.

Definition 2.1. A measurable function $G : \Gamma_0 \rightarrow \mathbb{R}$ is said have bounded support if: (a) there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that $G(\eta) = 0$ whenever $\eta \cap \Lambda^c \neq \emptyset$; (b) there exists $N \in \mathbb{N}_0$ such that $G(\eta) = 0$ whenever $|\eta| > N$. Here $\Lambda^c := \mathbb{R}^d \setminus \Lambda$ and $|\cdot|$ stands for cardinality. By $\Lambda(G)$ and $N(G)$ we denote the smallest Λ and N with the properties just mentioned. By $B_{bs}(\Gamma_0)$ we denote the set of all such functions.

The Lebesgue-Poisson measure λ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is defined by the following formula

$$\int_{\Gamma_0} G(\eta) \lambda(d\eta) = G(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n, \quad (2.3)$$

which has to hold for all $G \in B_{bs}(\Gamma_0)$. For $\gamma \in \Gamma$, by writing $\eta \in \gamma$ we mean that $\eta \subset \gamma$ is nonempty and finite. For $G \in B_{bs}(\Gamma)$, we set

$$(KG)(\gamma) = \sum_{\eta \in \gamma} G(\eta). \quad (2.4)$$

Note that the sum in (2.4) is finite and KG is a cylinder function on Γ . The latter means that it is $\mathcal{B}(\Gamma_{\Lambda(G)})$ -measurable, see Definition 2.1. Moreover,

$$|(KG)(\gamma)| \leq (1 + |\gamma \cap \Lambda(G)|)^{N(G)}. \quad (2.5)$$

2.2. Correlation functions. Like in (2.2), we introduce $k_\mu : \Gamma_0 \rightarrow \mathbb{R}$ such that $k_\mu(\eta) = k_\mu^{(n)}(x_1, \dots, x_n)$ for $\eta = \{x_1, \dots, x_n\}$, $n \in \mathbb{N}$. We also set $k_\mu(\emptyset) = 1$. With the help of the measure introduced in (2.3), the formulas for B_μ in (1.5) and (1.6) can be combined into the following formula

$$\begin{aligned} B_\mu(\theta) &= \int_{\Gamma_0} k_\mu(\eta) \prod_{x \in \eta} \theta(x) \lambda(d\eta) =: \int_{\Gamma_0} k_\mu(\eta) e(\eta; \theta) \lambda(d\eta) \\ &= \int_{\Gamma} \prod_{x \in \gamma} (1 + \theta(x)) \mu(d\gamma) =: \int_{\Gamma} F_\theta(\gamma) \mu(d\gamma). \end{aligned} \quad (2.6)$$

Thereby, we can transform the action of L on F , as in (1.2), to the action of L^Δ on k_μ according to the rule

$$\int_{\Gamma} (LF_\theta)(\gamma) \mu(d\gamma) = \int_{\Gamma_0} (L^\Delta k_\mu)(\eta) e(\eta; \theta) \lambda(d\eta). \quad (2.7)$$

This will allow us to pass from (1.1) to the corresponding Cauchy problem for the correlation functions, cf. (3.1) below. The main advantage here is that k_μ is a function of *finite* configurations.

For $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ and $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, let μ^Λ be as in (2.1). Then μ^Λ is absolutely continuous with respect to the restriction λ^Λ to $\mathcal{B}(\Gamma_\Lambda)$ of the measure defined in (2.3), and hence we may write

$$\mu^\Lambda(d\eta) = R_\mu^\Lambda(\eta) \lambda^\Lambda(d\eta), \quad \eta \in \Gamma_\Lambda. \quad (2.8)$$

Then the correlation function k_μ and the Radon-Nikodym derivative R_μ^Λ satisfy

$$k_\mu(\eta) = \int_{\Gamma_\Lambda} R_\mu^\Lambda(\eta \cup \xi) \lambda^\Lambda(d\xi). \quad (2.9)$$

Note that (2.9) relates R_μ^Λ with the restriction of k_μ to Γ_Λ . The fact that these are the restrictions of one and the same function $k_\mu : \Gamma_0 \rightarrow \mathbb{R}$ corresponds to the Kolmogorov consistency of the family $\{\mu^\Lambda : \Lambda \in \mathcal{B}(\mathbb{R}^d)\}$.

By (2.4), (2.1), and (2.8) we get

$$\int_{\Gamma} (KG)(\gamma) \mu(d\gamma) = \langle\langle G, k_\mu \rangle\rangle, \quad (2.10)$$

holding for each $G \in B_{\text{bs}}(\Gamma_0)$ and $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$. Here

$$\langle\langle G, k \rangle\rangle := \int_{\Gamma_0} G(\eta) k(\eta) \lambda(d\eta), \quad (2.11)$$

for suitable G and k . Define

$$B_{\text{bs}}^*(\Gamma_0) = \{G \in B_{\text{bs}}(\Gamma_0) : (KG)(\gamma) \geq 0 \text{ for all } \gamma \in \Gamma\}. \quad (2.12)$$

By [7, Theorems 6.1 and 6.2 and Remark 6.3] one can prove the next statement.

Proposition 2.2. *Let a measurable function $k : \Gamma_0 \rightarrow \mathbb{R}$ have the following properties:*

$$(a) \quad \langle\langle G, k \rangle\rangle \geq 0, \quad \text{for all } G \in B_{\text{bs}}^*(\Gamma_0); \quad (2.13)$$

$$(b) \quad k(\emptyset) = 1; \quad (c) \quad k(\eta) \leq C^{|\eta|},$$

with (c) holding for some $C > 0$ and λ -almost all $\eta \in \Gamma_0$. Then there exists a unique $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ for which k is the correlation function.

2.3. The model. The model we consider is specified by the operator L given in (1.2) with

$$c(x, y, \gamma) = a(x - y) \exp \left(- \sum_{z \in \gamma} \phi(y - z) \right). \quad (2.14)$$

The jump kernel $a : \mathbb{R}^d \rightarrow [0, +\infty)$ is such that $a(x) = a(-x)$ and

$$\int_{\mathbb{R}^d} a(x) dx =: \alpha < \infty, \quad (2.15)$$

whereas the repulsion potential $\phi : \mathbb{R}^d \rightarrow [0, +\infty)$, $\phi(x) = \phi(-x)$, is supposed to be such that

$$\int_{\mathbb{R}^d} \phi(x) dx =: \langle \phi \rangle < \infty, \quad \text{ess sup}_{x \in \mathbb{R}^d} \phi(x) =: \bar{\phi} < \infty. \quad (2.16)$$

Then also

$$\int_{\mathbb{R}^d} \left(1 - \exp(-\phi(x))\right) dx \leq \langle \phi \rangle. \quad (2.17)$$

By (1.2) and (2.7) one obtains, cf. [2, Eq. (3.1)],

$$\begin{aligned} (L^\Delta k)(\eta) &= \sum_{y \in \eta} \int_{\mathbb{R}^d} a(x-y) e(\tau_y; \eta \setminus y \cup x) (Q_y k)(\eta \setminus y \cup x) dx \\ &\quad - \sum_{x \in \eta} \int_{\mathbb{R}^d} a(x-y) e(\tau_y; \eta) (Q_y k)(\eta) dy. \end{aligned} \quad (2.18)$$

Here e is as in (2.6),

$$(Q_y k)(\eta) := \int_{\Gamma_0} k(\eta \cup \xi) e(t_y; \xi) \lambda(d\xi), \quad (2.19)$$

and

$$\tau_x(y) := \exp(-\phi(x-y)), \quad t_x(y) := \tau_x(y) - 1, \quad x, y \in \mathbb{R}^d. \quad (2.20)$$

3. THE RESULT

As mentioned above, instead of directly dealing with the problem in (1.3) we pass from μ_0 to the corresponding correlation function k_{μ_0} and then consider the problem

$$\frac{d}{dt} k_t = L^\Delta k_t, \quad k_t|_{t=0} = k_{\mu_0} \quad (3.1)$$

with L^Δ given in (2.18). The aim is to prove the existence of a unique global solution k_t of (3.1) which is the correlation function of a unique state $\mu_t \in \mathcal{P}_{\text{exp}}(\Gamma)$.

We begin by defining (3.1) in the corresponding spaces of functions $k : \Gamma_0 \rightarrow \mathbb{R}$. From the very representation (1.6), see also (2.6), it follows that $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ implies

$$|k_\mu(\eta)| \leq C \exp(\vartheta |\eta|),$$

holding for λ -almost all $\eta \in \Gamma_0$, some $C > 0$, and $\vartheta \in \mathbb{R}$. Keeping this in mind we set

$$\|k\|_\vartheta = \text{ess sup}_{\eta \in \Gamma_0} \{ |k_\mu(\eta)| \exp(-\vartheta |\eta|) \}. \quad (3.2)$$

Then

$$\mathcal{K}_\vartheta := \{k : \Gamma_0 \rightarrow \mathbb{R} : \|k\|_\vartheta < \infty\}$$

is a Banach space with norm (3.2) and the usual linear operations. In fact, we are going to use the ascending scale of such spaces \mathcal{K}_ϑ , $\vartheta \in \mathbb{R}$, with the property

$$\mathcal{K}_\vartheta \hookrightarrow \mathcal{K}_{\vartheta'}, \quad \vartheta < \vartheta', \quad (3.3)$$

where \hookrightarrow denotes continuous embedding. Set, cf. (2.10), (2.11), and (2.12),

$$\mathcal{K}_\vartheta^* = \{k \in \mathcal{K}_\vartheta : \langle\langle G, k \rangle\rangle \geq 0 \text{ for all } G \in B_{\text{bs}}^*(\Gamma_0)\}, \quad (3.4)$$

which is a subset of the cone

$$\mathcal{K}_\vartheta^+ = \{k \in \mathcal{K}_\vartheta : k(\eta) \geq 0 \text{ for } \lambda - \text{almost all } \eta \in \Gamma_0\}. \quad (3.5)$$

By Proposition 2.2 it follows that each $k \in \mathcal{K}_\vartheta^*$ such that $k(\emptyset) = 1$ is the correlation function of a unique state $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$. Then we define

$$\mathcal{K} = \bigcup_{\vartheta \in \mathbb{R}} \mathcal{K}_\vartheta, \quad \mathcal{K}^* = \bigcup_{\vartheta \in \mathbb{R}} \mathcal{K}_\vartheta^*. \quad (3.6)$$

As a sum of Banach spaces, the linear space \mathcal{K} is equipped with the corresponding inductive topology which turns it into a locally convex space.

For a given $\vartheta \in \mathbb{R}$, by (2.18) – (2.20) we define L_ϑ^Δ as a linear operator in \mathcal{K}_ϑ with domain

$$\mathcal{D}(L_\vartheta^\Delta) = \{k \in \mathcal{K}_\vartheta : L^\Delta k \in \mathcal{K}_\vartheta\}. \quad (3.7)$$

Lemma 3.1. *For each $\vartheta'' < \vartheta$, cf. (3.3), it follows that $\mathcal{K}_{\vartheta''} \subset \mathcal{D}(L_\vartheta^\Delta)$.*

Proof. For $\vartheta'' < \vartheta$, by (2.17), (2.19), (2.20), and (3.2) we have

$$\begin{aligned} |(Q_y k)(\eta)| &\leq \|k\|_{\vartheta''} \exp(\vartheta'' |\eta|) \\ &\times \int_{\Gamma_0} \exp(\vartheta'' |\xi|) \prod_{z \in \xi} \left(1 - \exp(-\phi(z - y))\right) \lambda(d\xi) \\ &\leq \|k\|_{\vartheta''} \exp(\vartheta'' |\eta|) \exp(\langle \phi \rangle e^{\vartheta''}). \end{aligned}$$

Now we apply the latter estimate and (2.15) in (2.18) and obtain

$$|(L^\Delta k)(\eta)| \leq 2\alpha \|k\|_{\vartheta''} \exp(\vartheta'' |\eta|) |\eta| \exp(\langle \phi \rangle e^{\vartheta''}). \quad (3.8)$$

By means of the inequality $x \exp(-\sigma x) \leq 1/e\sigma$, $x, \sigma > 0$, we get from (3.2) and (3.8) the following estimate

$$\|L^\Delta k\|_\vartheta \leq \frac{2\alpha \|k\|_{\vartheta''}}{e(\vartheta - \vartheta'')} \exp(\langle \phi \rangle e^{\vartheta''}), \quad (3.9)$$

which yields the proof. \square

Corollary 3.2. *For each $\vartheta, \vartheta'' \in \mathbb{R}$ such that $\vartheta'' < \vartheta$, the expression in (2.18) defines a bounded linear operator $L_{\vartheta\vartheta''}^\Delta : \mathcal{K}_{\vartheta''} \rightarrow \mathcal{K}_\vartheta$ the norm of which can be estimated by means of (3.9).*

In what follows, we consider two types of operators defined by the expression in (2.18): (a) unbounded operators $(L_\vartheta^\Delta, \mathcal{D}(L_\vartheta^\Delta))$, $\vartheta \in \mathbb{R}$, with domains as in (3.7) and Lemma 3.1; (b) bounded operators $L_{\vartheta\vartheta''}^\Delta$, as in Corollary 3.2. These operators are related to each other in the following way:

$$\forall \vartheta'' < \vartheta \quad \forall k \in \mathcal{K}_{\vartheta''} \quad L_{\vartheta\vartheta''}^\Delta k = L_\vartheta^\Delta k. \quad (3.10)$$

By means of the bounded operators $L_{\vartheta\vartheta''}^\Delta : \mathcal{K}_{\vartheta''} \rightarrow \mathcal{K}_\vartheta$ we also define a continuous linear operator $L^\Delta : \mathcal{K} \rightarrow \mathcal{K}$, see (3.6). In view of this, we consider the following two equations. The first one is

$$\frac{d}{dt} k_t = L_\vartheta^\Delta k_t, \quad k_t|_{t=0} = k_{\mu_0}, \quad (3.11)$$

considered as an equation in a given Banach space \mathcal{K}_ϑ . The second equation is (3.1) with L^Δ given in (2.18) considered in the locally convex space \mathcal{K} .

Definition 3.3. By a solution of (3.11) on a time interval, $[0, T)$, $T \leq +\infty$, we mean a continuous map $[0, T) \ni t \mapsto k_t \in \mathcal{D}(L_\vartheta^\Delta)$ such that the map $[0, T) \ni t \mapsto dk_t/dt \in \mathcal{K}_\vartheta$ is also continuous and both equalities in (3.11) are satisfied. Likewise, a continuously differentiable map $[0, T) \ni t \mapsto k_t \in \mathcal{K}$ is said to be a solution of (3.1) in \mathcal{K} if both equalities therein are satisfied for all t . Such a solution is called global if $T = +\infty$.

Remark 3.4. The map $[0, T) \ni t \mapsto k_t \in \mathcal{K}$ is a solution of (3.1) if and only if, for each $t \in [0, T)$, there exists $\vartheta'' \in \mathbb{R}$ such that $k_t \in \mathcal{K}_{\vartheta''}$ and, for each $\vartheta > \vartheta''$, the map $t \mapsto k_t$ is continuously differentiable at t in \mathcal{K}_ϑ and $dk_t/dt = L_\vartheta^\Delta k_t = L_{\vartheta\vartheta''}^\Delta k_t$.

Our main result is contained in the following statement.

Theorem 3.5. *Assume that (2.15) and (2.16) hold. Then for each $\mu_0 \in \mathcal{P}_{\text{exp}}(\Gamma)$, the problem (3.1) with $k_0 = k_{\mu_0}$ has a unique global solution $k_t \in \mathcal{K}^* \subset \mathcal{K}$ which has the property $k_t(\emptyset) = 1$. Therefore, for each $t \geq 0$ there exists a unique state $\mu_t \in \mathcal{P}_{\text{exp}}(\Gamma)$ such that $k_t = k_{\mu_t}$. Moreover, let k_0 and $C > 0$ be such that $k_0(\eta) \leq C^{|\eta|}$ for λ -almost all $\eta \in \Gamma_0$, see (2.13). Then the mentioned solution satisfies*

$$\forall t \geq 0 \quad 0 \leq k_t(\eta) \leq C^{|\eta|} \exp(t\alpha|\eta|). \quad (3.12)$$

4. THE PROOF OF THEOREM 3.5

Our strategy of the proof resembles that used in [6]. Basically, it consist in performing the following three steps: (a) proving the existence of a unique solution of (3.11) with $t < T$ for some $T < \infty$; (b) proving the identification lemma, i.e., that the solution of (3.11) satisfies the conditions of Proposition 2.2 and hence is the correlation function of a unique sub-Poissonian state; (c) constructing the extension of the solution to all $t > 0$ by employing the positive definiteness obtained in (b).

4.1. Finite time horizon. For $\vartheta, \vartheta' \in \mathbb{R}$ such that $\vartheta < \vartheta'$, we set, cf. (3.9),

$$T(\vartheta', \vartheta) = \frac{\vartheta' - \vartheta}{2\alpha} \exp\left(-\langle \phi \rangle e^{\vartheta'}\right). \quad (4.1)$$

For a fixed $\vartheta' \in \mathbb{R}$, $T(\vartheta', \vartheta)$ can be made as big as one wants by taking small enough ϑ . However, if ϑ is fixed, then

$$\sup_{\vartheta' > \vartheta} T(\vartheta', \vartheta) = \frac{\delta(\vartheta)}{2\alpha} \exp\left(-\frac{1}{\delta(\vartheta)}\right) =: \tau(\vartheta) < \infty, \quad (4.2)$$

where $\delta(\vartheta)$ is the unique positive solution of the equation

$$\delta e^\delta = \exp(-\vartheta - \log \langle \phi \rangle). \quad (4.3)$$

Remark 4.1. The supremum in (4.2) is attained at $\vartheta' = \vartheta + \delta(\vartheta)$. Note also that $\delta(\vartheta) \rightarrow 0$, and hence $\tau(\vartheta) \rightarrow 0$, as $\vartheta \rightarrow +\infty$.

Lemma 4.2. *For an arbitrary $\vartheta \in \mathbb{R}$, the problem in (3.11) with $k_0 \in \mathcal{K}_\vartheta$ has a unique solution $k_t \in \mathcal{K}_{\vartheta+\delta(\vartheta)}$ on the time interval $[0, \tau(\vartheta))$.*

Proof. Take $T < \tau(\vartheta)$ and then pick $\vartheta' \in (\vartheta, \vartheta + \delta(\vartheta))$ such that $T < T(\vartheta', \vartheta)$. Let $\mathcal{L}(\mathcal{K}_\vartheta, \mathcal{K}_{\vartheta'})$ stand for the Banach space of bounded linear operators acting from \mathcal{K}_ϑ to $\mathcal{K}_{\vartheta'}$ equipped with the corresponding operator norm. Our aim is to construct the family

$$S_{\vartheta', \vartheta}(t) \in \mathcal{L}(\mathcal{K}_\vartheta, \mathcal{K}_{\vartheta'}), \quad t \in [0, T(\vartheta', \vartheta)), \quad (4.4)$$

defined by the series

$$S_{\vartheta', \vartheta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (L^\Delta)_{\vartheta', \vartheta}^n. \quad (4.5)$$

In (4.5), $(L^\Delta)_{\vartheta', \vartheta}^0$ is the embedding operator and

$$(L^\Delta)_{\vartheta', \vartheta}^n := \prod_{l=1}^n L_{\vartheta_l, \vartheta_{l-1}}^\Delta, \quad \vartheta_l = \vartheta + l(\vartheta' - \vartheta)/n, \quad (4.6)$$

for $n \in \mathbb{N}$. Now we take into account that $\vartheta_l - \vartheta_{l-1} = (\vartheta' - \vartheta)/n$ and that L^Δ satisfies (3.9). Then we get

$$\|L_{\vartheta_l, \vartheta_{l-1}}^\Delta\| \leq \left(\frac{n}{e}\right) (\vartheta' - \vartheta) \left\{2\alpha \exp\left(\langle \phi \rangle e^{\vartheta'}\right)\right\}^{-1} \leq n/eT(\vartheta', \vartheta), \quad (4.7)$$

see (3.9) and (4.1). Next we apply (4.7) in (4.6) and conclude that the series in (4.5) converges in the operator norm, uniformly on $[0, T]$, to the operator-valued function $[0, T] \ni t \mapsto S_{\vartheta', \vartheta}(t) \in \mathcal{L}(\mathcal{K}_\vartheta, \mathcal{K}_{\vartheta'})$ such that

$$\forall t \in [0, T] \quad \|S_{\vartheta', \vartheta}(t)\| \leq \frac{T(\vartheta', \vartheta)}{T(\vartheta', \vartheta) - t}. \quad (4.8)$$

Likewise, for $\vartheta'' \in (\vartheta', \vartheta + \delta(\vartheta)]$, we get

$$\begin{aligned} \frac{d}{dt} S_{\vartheta''\vartheta}(t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} (L^\Delta)^{n+1}_{\vartheta''\vartheta} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} L^\Delta_{\vartheta''\vartheta'} (L^\Delta)^n_{\vartheta'\vartheta} = L^\Delta_{\vartheta''\vartheta'} S_{\vartheta'\vartheta}(t), \quad t \in [0, T] \end{aligned} \quad (4.9)$$

Then

$$k_t = S_{\vartheta'\vartheta}(t)k_0 \in \mathcal{K}_{\vartheta'} \subset \mathcal{D}(L^\Delta_{\vartheta''}), \quad (4.10)$$

see Lemma 3.1, is a solution of (3.11) on the time interval $[0, \tau(\vartheta))$ since $T < \tau(\vartheta)$ has been taken in an arbitrary way.

Let us prove that the solution given in (4.10) is unique. In view of the linearity, to this end it is enough to show that the problem in (3.11) with the zero initial condition has a unique solution. Assume that $v_t \in \mathcal{D}(L^\Delta_{\vartheta+\delta(\vartheta)})$ is one of the solutions. Then v_t lies in $\mathcal{K}_{\vartheta''}$ for each $\vartheta'' > \vartheta + \delta(\vartheta)$, see (3.3). Fix any such ϑ'' and then take $t < \tau(\vartheta)$ such that $t < T(\vartheta'', \vartheta + \delta(\vartheta))$. Then, cf. (3.10),

$$\begin{aligned} v_t &= \int_0^t L^\Delta_{\vartheta''\bar{\vartheta}} v_s ds \\ &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} (L^\Delta)^n_{\vartheta''\bar{\vartheta}} v_{t_n} dt_n \cdots dt_1, \end{aligned}$$

where $\bar{\vartheta} := \vartheta + \delta(\vartheta)$ and $n \in \mathbb{N}$ is an arbitrary number. Similarly as above we get from the latter

$$\|v_t\|_{\vartheta''} \leq \frac{t^n}{n!} \left(\frac{n}{eT(\vartheta'', \bar{\vartheta})} \right)^n \sup_{s \in [0, t]} \|v_s\|_{\bar{\vartheta}}.$$

Since n is an arbitrary number, this yields $v_s = 0$ for all $s \in [0, t]$. The extension of this result to all $t < \tau(\vartheta)$ can be done by repeating this procedure due times. \square

Remark 4.3. Similarly as in obtaining (4.9) we have that for all $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathbb{R}$ such that $\vartheta_0 < \vartheta_1 < \vartheta_2$, the following holds

$$S_{\vartheta_2\vartheta_0}(t+s) = S_{\vartheta_2\vartheta_1}(t) S_{\vartheta_1\vartheta_0}(s), \quad (4.11)$$

$$t \in [0, T(\vartheta_2, \vartheta_1)), \quad s \in [0, T(\vartheta_1, \vartheta_0)).$$

4.2. The identification lemma. Here we show that the solution of (3.11) given in (4.10) has the property $k_t \in \mathcal{K}_{\vartheta}^*$, see (3.4). To some extent, we follow the way of proving Theorem 3.7 in [2]. However, due to an elegant argument provided by the Denjoy-Carleman theorem [3], the present proof is more complete and transparent.

Lemma 4.4. *Let ϑ^* be as in Corollary 3.2. Then for each $t \in [0, T(\vartheta, \vartheta^*))$, the operator defined in (4.5) has the property*

$$S_{\vartheta\vartheta^*}(t) : \mathcal{K}_{\vartheta^*}^* \rightarrow \mathcal{K}_{\vartheta}^*. \quad (4.12)$$

Proof. Let $\mu_0 \in \mathcal{P}_{\text{exp}}(\Gamma)$ be such that $k_{\mu_0} \in \mathcal{K}_{\vartheta^*}^*$, see Proposition 2.2. For $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, let μ_0^Λ and $R_{\mu_0}^\Lambda$ be as in (2.8). For $N \in \mathbb{N}$, we then set

$$R_0^{\Lambda, N}(\eta) = R_{\mu_0}^\Lambda(\eta) I_N(\eta), \quad \eta \in \Gamma_0, \quad (4.13)$$

where $I_N(\eta) = 1$ whenever $|\eta| \leq N$ and $I_N(\eta) = 0$ otherwise. Set

$$\mathcal{R} = L^1(\Gamma_0, d\lambda), \quad \mathcal{R}_\beta = L^1(\Gamma_0, b_\beta d\lambda), \quad (4.14)$$

$$b_\beta(\eta) := \exp\left(\beta|\eta|\right), \quad \beta > 0.$$

Let $\|\cdot\|_{\mathcal{R}}$ and $\|\cdot\|_{\mathcal{R}_\beta}$ be the norms of the spaces introduced in (4.14) and \mathcal{R}^+ and \mathcal{R}_β^+ be the corresponding cones of positive elements. For each $\beta > 0$, $R_0^{\Lambda, N}$ defined in (4.13) lies in $\mathcal{R}_\beta^+ \subset \mathcal{R}^+$ and is such that $\|R_0^{\Lambda, N}\|_{\mathcal{R}} \leq 1$. By means of perturbative methods developed in [10], see [2, Section 3.2], it is possible to show that L^* related by (1.4) to L given in (1.2) generates the evolution of states $\mu_0 \mapsto \mu_t$, $t \geq 0$, whenever μ_0 has the property $\mu_0(\Gamma_0) = 1$, which is the case for μ_0^Λ . Moreover, for each $t \geq 0$, the mentioned μ_t is absolutely continuous with respect to λ , and the equation for $R_t = d\mu_t/d\lambda$ corresponding to (1.3) can be written in the form

$$\frac{d}{dt} R_t = L^\dagger R_t, \quad R_t|_{t=0} = R_{\mu_0}, \quad (4.15)$$

where, cf. (2.18), L^\dagger is defined by the relation $L^\dagger R = d(L^* \mu)/d\lambda$, and hence acts according to the following formula

$$(L^\dagger R)(\eta) = \sum_{y \in \eta} \int_{\mathbb{R}^d} a(x-y) e(\tau_y; \eta) R(\eta \setminus y \cup x, \eta) dx - \Psi(\eta) R(\eta),$$

$$\Psi(\eta) := \sum_{x \in \eta} \int_{\mathbb{R}^d} a(x-y) e(\tau_y; \eta) dy. \quad (4.16)$$

Like in the proof of [2, Theorem 3.7], one shows that L^\dagger generates a stochastic C_0 -semigroup, $S_R := \{S_R(t)\}_{t \geq 0}$, on \mathcal{R} , which leaves invariant each \mathcal{R}_β , $\beta > 0$. Then the solution of (4.15) is $R_t = S_R(t) R_0$. For $R_0^{\Lambda, N}$ as in (4.13), we then set

$$R_t^{\Lambda, N}(t) = S_R(t) R_0^{\Lambda, N}, \quad t > 0. \quad (4.17)$$

Then $R_t^{\Lambda, N} \in \mathcal{R}_\beta^+ \subset \mathcal{R}^+$ and $\|R_t^{\Lambda, N}\|_{\mathcal{R}} \leq 1$. This yields that, for each $G \in B_{\text{bs}}^*(\Gamma_0)$, see (2.11) and (2.12), the following holds

$$\langle\langle KG, R_t^{\Lambda, N} \rangle\rangle \geq 0, \quad t \geq 0. \quad (4.18)$$

The integral in (4.18) exists as $R_t^{\Lambda, N} \in \mathcal{R}_\beta$ and KG satisfies (2.5). Moreover, like in (3.9), for each β' such that $0 < \beta' < \beta$, we derive from (4.2) the following estimate

$$\|L^\dagger R\|_{\mathcal{R}_{\beta'}} \leq \frac{2\alpha\|R\|_{\mathcal{R}_\beta}}{e(\beta - \beta')}.$$

This allows us to define the corresponding bounded operators $(L^\dagger)_{\beta'\beta}^n : \mathcal{R}_\beta \rightarrow \mathcal{R}_{\beta'}$, $n \in \mathbb{N}$, cf. (4.6), the norms of which satisfy

$$\|(L^\dagger)_{\beta'\beta}^n\| \leq n^n (e\bar{T}(\beta, \beta'))^{-n}. \quad (4.19)$$

On the other hand, we have that, cf. (2.9) and (4.13),

$$k_0^{\Lambda, N}(\eta) := \int_{\Gamma_0} R_0^{\Lambda, N}(\eta \cup \xi) \lambda(d\xi) \quad (4.20)$$

is such that $k_0^{\Lambda, N} \in \mathcal{K}_{\vartheta^*}^\star$, and hence we may get

$$k_t^{\Lambda, N} = S_{\vartheta\vartheta^*}(t)k_0^{\Lambda, N}, \quad t \in [0, T(\vartheta, \vartheta^*)), \quad (4.21)$$

where $S_{\vartheta\vartheta^*}(t)$ is given in (4.5). Then the proof of (4.12) consists in showing:

$$(i) \quad \forall G \in B_{\text{bs}}^\star(\Gamma_0) \quad \langle\langle G, k_t^{\Lambda, N} \rangle\rangle \geq 0; \quad (4.22)$$

$$(ii) \quad \langle\langle G, S_{\vartheta\vartheta^*}^1(t)k_0 \rangle\rangle = \lim_{\Lambda \rightarrow \mathbb{R}^d} \lim_{N \rightarrow +\infty} \langle\langle G, k_t^{\Lambda, N} \rangle\rangle.$$

To prove claim (i) of (4.22) for $G \in B_{\text{bs}}^\star(\Gamma_0)$, cf. (2.12), we set

$$\varphi_G(t) = \langle\langle KG, R_t^{\Lambda, N} \rangle\rangle, \quad \psi_G(t) = \langle\langle G, k_t^{\Lambda, N} \rangle\rangle, \quad (4.23)$$

where ψ_G is defined for t as in (4.21). For a given $t \in (0, T(\vartheta, \vartheta^*))$, we pick $\vartheta' < \vartheta$ such that $t < T(\vartheta', \vartheta^*)$, and hence $k_s^{\Lambda, N} \in \mathcal{K}_{\vartheta'}$ for $s \in [0, t]$. Then the direct calculation based on (4.9) yields for the n -th derivative

$$\psi_G^{(n)}(t) = \langle\langle G, (L^\Delta)_{\vartheta\vartheta'}^n k_t^{\Lambda, N} \rangle\rangle, \quad n \in \mathbb{N}.$$

As in obtaining (4.8) we then get from the latter

$$|\psi_G^{(n)}(t)| \leq A^n n^n C_{\vartheta'}(G) \sup_{s \in [0, t]} \|k_s^{\Lambda, N}\|_{\vartheta'}. \quad (4.24)$$

Here $A = 1/eT(\vartheta, \vartheta')$ and

$$C_{\vartheta'}(G) = \int_{\Gamma_0} |G(\eta)| \exp(\vartheta'|\eta|) \lambda(d\eta) < \infty,$$

as $G \in B_{\text{bs}}(\Gamma_0)$, see Definition 2.1. Likewise, from (4.17) we get

$$\varphi_G^{(n)}(t) = \langle\langle KG, (L^\dagger)_{\beta'\beta}^n R_t^{\Lambda, N} \rangle\rangle$$

For the same t as in (4.24), by (4.19) we have from the latter

$$|\varphi_G^{(n)}(t)| \leq \bar{A}^n n^n C_{\beta'}(G) \sup_{s \in [0, t]} \|R_s^{\Lambda, N}\|_{\beta'}. \quad (4.25)$$

Here $\bar{A} = 1/e\bar{T}(\beta', \beta)$ and

$$C_{\beta'}(G) = \operatorname{ess\,sup}_{\eta \in \Gamma_0} |KG(\eta)| \exp(-\beta'|\eta|) < \infty,$$

which holds in view of (2.5). By (2.18) and (4.20) it follows that

$$(L^\Delta k_0^{\Lambda, N})(\eta) = \int_{\Gamma_0^2} (L^\dagger R_0^{\Lambda, N})(\eta \cup \xi) \lambda(d\xi),$$

which then yields

$$\forall n \in \mathbb{N}_0 \quad \varphi_G^{(n)}(0) = \psi_G^{(n)}(0). \quad (4.26)$$

By (4.25) and (4.24) both functions defined in (4.23) are quasi-analytic on $[0, t]$. Then by the Denjoy-Carleman theorem [3], (4.26) implies, see (4.18),

$$\forall t \in [0, T(\vartheta, \vartheta^*)) \quad \psi_G(t) = \varphi_G(t) \geq 0, \quad (4.27)$$

which yields the first line in (4.22). The convergence in claim (ii) of (4.22) is proved in a standard way, see Appendix in [2]. \square

Note that (4.27) yields also that

$$\forall t \in [0, T(\vartheta, \vartheta^*)) \quad \langle\langle G, q_t^{\Lambda, N} \rangle\rangle = \langle\langle G, k_t^{\Lambda, N} \rangle\rangle, \quad (4.28)$$

where G and $k_t^{\Lambda, N}$ are as in (4.23) and

$$q_t^{\Lambda, N}(\eta) := \int_{\Gamma_0^2} R_t^{\Lambda, N}(\eta \cup \xi) \lambda(d\xi). \quad (4.29)$$

4.3. An auxiliary evolution. The evolution which we construct now will be used to extending the solution k_t given in (4.10) to the global solution as stated in Theorem 3.5. The construction employs the operator

$$(\bar{L}k)(\eta) = \sum_{y \in \eta} \int_{\mathbb{R}^d} a(x - y) k(\eta \setminus y \cup x) dx \quad (4.30)$$

obtained from L^Δ given in (2.18) by putting $\phi = 0$, and then dropping the second term. Hence, like in (3.9) we get

$$\|\bar{L}k\|_\vartheta \leq \frac{2\alpha \|k\|_{\vartheta''}}{e(\vartheta - \vartheta'')}, \quad (4.31)$$

which allows us to introduce the operators $(\bar{L}_\vartheta, \mathcal{D}(\bar{L}_\vartheta))$ and $\bar{L}_{\vartheta\vartheta''} \in \mathcal{L}(\mathcal{K}_{\vartheta''}, \mathcal{K}_\vartheta)$ such that, cf. (3.10),

$$\forall k \in \vartheta'' \quad \bar{L}_{\vartheta\vartheta''} k = \bar{L}_\vartheta k, \quad \vartheta'' < \vartheta.$$

Like above, we have that

$$\mathcal{K}_{\vartheta''} \subset \mathcal{D}(\bar{L}_\vartheta) := \{k \in \mathcal{K}_\vartheta : \bar{L}k \in \mathcal{K}_\vartheta\}, \quad \vartheta'' < \vartheta.$$

Note that

$$\bar{L}_{\vartheta\vartheta''} : \mathcal{K}_{\vartheta''}^+ \rightarrow \mathcal{K}_\vartheta^+, \quad \vartheta'' < \vartheta, \quad (4.32)$$

see (3.5). For $n \in \mathbb{N}$, we define $(\bar{L})_{\vartheta'\vartheta}^n$ similarly as in (4.6) and denote, cf. (4.1),

$$\bar{T}(\vartheta', \vartheta) = (\vartheta' - \vartheta)/2\alpha, \quad \vartheta < \vartheta'. \quad (4.33)$$

Our aim is to study the operator valued function defined by the series

$$\bar{S}_{\vartheta'\vartheta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\bar{L})_{\vartheta'\vartheta}^n. \quad (4.34)$$

Lemma 4.5. *For each $\vartheta_0, \vartheta \in \mathbb{R}$ such that $\vartheta_0 < \vartheta$, the series in (4.34) defines a continuous function*

$$[0, \bar{T}(\vartheta, \vartheta_0)) \ni t \mapsto \bar{S}_{\vartheta\vartheta_0}(t) \in \mathcal{L}(\mathcal{K}_{\vartheta_0}, \mathcal{K}_{\vartheta}), \quad (4.35)$$

which has the following properties:

- (a) For t as in (4.35), let $\vartheta'' \in (\vartheta_0, \vartheta)$ be such that $t < \bar{T}(\vartheta'', \vartheta_0)$. Then, cf. (4.9),

$$\frac{d}{dt} \bar{S}_{\vartheta\vartheta_0}(t) = \bar{L}_{\vartheta\vartheta''} \bar{S}_{\vartheta''\vartheta_0}(t). \quad (4.36)$$

- (b) The problem

$$\frac{d}{dt} u_t = \bar{L}_{\vartheta} u_t, \quad u_t|_{t=0} = u_0 \in \mathcal{K}_{\vartheta_0}^+, \quad (4.37)$$

has a unique solution $u_t \in \mathcal{K}_{\vartheta}^+$ on the time interval $[0, \bar{T}(\vartheta, \vartheta_0))$ given by

$$u_t = \bar{S}_{\vartheta''\vartheta_0}(t) u_0, \quad (4.38)$$

where, for a fixed $t \in [0, \bar{T}(\vartheta, \vartheta_0))$, ϑ'' is chosen to satisfy $t < \bar{T}(\vartheta'', \vartheta_0)$.

Proof. Proceeding as in the proof of Lemma 4.2, by means of the estimate in (4.31) we prove the convergence of the series in (4.34). This allows also for proving (4.36), which yields the existence of the solution of (4.37) in the form given in (4.38). The uniqueness is proved analogously as in the case of Lemma 4.2. The stated positivity of u_t follows from (4.34) and (4.32). \square

Corollary 4.6. *For a given $C > 0$, we let in (4.37) and (4.38) $\vartheta_0 = \log C$ and $u_0(\eta) = C^{|\eta|}$. Then the unique solution of (4.37) is*

$$u_t(\eta) = C^{|\eta|} \exp \{t(\alpha|\eta|)\}. \quad (4.39)$$

This solution can naturally be continued to all $t > 0$ for which it lies in $\mathcal{K}_{\vartheta(t)}$ with

$$\vartheta(t) = \log C + t\alpha. \quad (4.40)$$

Proof. In view of the lack of interaction in (4.30), the equations for particular $u_t^{(n)}$ take the following form

$$\begin{aligned} \frac{d}{dt} u_t^{(n)}(x_1, \dots, x_{n_0}) = \\ \sum_{i=1}^n \int_{\mathbb{R}^d} a(x - x_i) u_t^{(n)}(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) dx \quad n \in \mathbb{N}, \end{aligned}$$

which for the initial translation invariant u_0 yields (4.39). \square

4.4. The global solution. As follows from Lemmas 4.2 and 4.4, the unique solution of the problem (3.11) with $k_0 \in \mathcal{K}_{\vartheta^*}^*$ lies in $\mathcal{K}_{\vartheta}^*$ for $t \in (0, T(\vartheta, \vartheta^*))$. At the same time, for fixed ϑ^* , $T(\vartheta, \vartheta^*)$ is bounded, see (4.2). This means that the mentioned solution cannot be directly continued as stated in Theorem 3.5. In this subsection, by a comparison method we prove that, for $t \in (0, T(\vartheta, \vartheta^*))$, k_t satisfies (3.12) which is then used to get the continuation in question, cf. Corollary 4.6. Recall that the operator Q_y , was introduced in (2.19) and the cone $\mathcal{K}_{\vartheta}^+$ was defined in (3.5).

Lemma 4.7. *For each $k_0 \in \mathcal{K}_{\vartheta^*}^*$ and $t \in (0, T(\vartheta, \vartheta^*))$, $k_t := S_{\vartheta\vartheta^*}(t)k_0$ has the property*

$$[k_t - e(\tau_y; \cdot)(Q_y k_t)] \in \mathcal{K}_{\vartheta}^+,$$

holding for Lebesgue-almost all $y \in \mathbb{R}^d$.

Proof. For a fixed y , we denote

$$v_{t,1} = k_t - Q_y k_t, \quad v_{t,2} = [1 - e(\tau_y; \cdot)]Q_y k_t.$$

The proof will be done if we show that, for all $G \in B_{\text{bs}}(\Gamma_0)$ such that $G(\eta) \geq 0$ for λ -almost all $\eta \in \Gamma_0$, the following holds

$$\langle\langle G, v_{t,j} \rangle\rangle \geq 0, \quad j = 1, 2. \quad (4.41)$$

Let Λ , N , and $k_0^{\Lambda, N}$ be as in (4.20), and then $k_t^{\Lambda, N}$ be as in (4.21). Next, let $v_{t,j}^{\Lambda, N}$, $j = 1, 2$, be defined as above with k_t replaced by $k_t^{\Lambda, N}$. By (4.28) and (4.29) we then get

$$\begin{aligned} \langle\langle G, Q_y k_t^{\Lambda, N} \rangle\rangle &= \int_{\Gamma_0} \tilde{G}(\eta) k_t^{\Lambda, N}(\eta) \lambda(d\eta) \\ &= \int_{\Gamma_0} \int_{\Gamma_0} \tilde{G}(\eta) R_t^{\Lambda, N}(\eta \cup \xi) \lambda(d\eta) \lambda(d\xi), \end{aligned} \quad (4.42)$$

where

$$\tilde{G}(\eta) := \sum_{\xi \subset \eta} e(t_y; \xi) G(\eta \setminus \xi).$$

Furthermore, by (4.42) we get

$$\begin{aligned} \langle\langle G, Q_y k_t^{\Lambda, N} \rangle\rangle &= \int_{\Gamma_0} G(\eta) \int_{\Gamma_0} \left(\int_{\Gamma_0} e(t_y; \zeta) R_t^{\Lambda, N}(\eta \cup \xi \cup \zeta) \lambda(d\zeta) \right) \lambda(d\eta) \lambda(d\xi) \\ &= \int_{\Gamma_0} G(\eta) \int_{\Gamma_0} \left(\sum_{\zeta \subset \xi} e(t_y; \zeta) \right) R_t^{\Lambda, N}(\eta \cup \xi) \lambda(d\eta) \lambda(d\xi). \end{aligned} \quad (4.43)$$

By (2.20) we have that

$$\sum_{\zeta \subset \xi} e(t_y; \zeta) = e(\tau_y; \xi).$$

We apply this in the last line of (4.43) and obtain

$$\begin{aligned}
& \langle\langle G, Q_y k_t^{\Lambda, N} \rangle\rangle \\
&= \int_{\Gamma_0} G(\eta) \int_{\Gamma_0} e(\tau_y; \xi) R_t^{\Lambda, N}(\eta \cup \xi) \lambda(d\eta) \lambda(d\xi) \\
&\leq \int_{\Gamma_0} G(\eta) \int_{\Gamma_0} R_t^{\Lambda, N}(\eta \cup \xi) \lambda(d\eta) \lambda(d\xi) \\
&= \langle\langle G, k_t^{\Lambda, N} \rangle\rangle,
\end{aligned} \tag{4.44}$$

which after the limiting transition as in (4.22) yields (4.41) for $j = 1$. For the same G , we set $\bar{G} = e(\tau_y; \cdot)G$. Then by (2.20) and the second line in (4.44) we get

$$\langle\langle \bar{G}, Q_y k_t^{\Lambda, N} \rangle\rangle \leq \langle\langle G, Q_y k_t^{\Lambda, N} \rangle\rangle,$$

which after the limiting transition as in (4.22) yields (4.41) for $j = 2$. \square

Lemma 4.8. *Let $C > 0$ be such that the initial condition in (3.11) satisfies $k_{\mu_0}(\eta) = k_0(\eta) \leq C^{|\eta|}$. Then for all $t < T(\vartheta, \vartheta^*)$ with $\vartheta^* = \log C$ and any $\vartheta > \vartheta^*$, the unique solution of (3.11) given by the formula*

$$k_t = S_{\vartheta\vartheta^*}(t)k_0 \tag{4.45}$$

satisfies (3.12) for λ -almost all $\eta \in \Gamma_0$.

Proof. Take any $\vartheta > \vartheta^*$ and fix $t < T(\vartheta, \vartheta^*)$; then pick $\vartheta^1 \in (\vartheta^*, \vartheta)$ such that $t < T(\vartheta^1, \vartheta^*)$. Next take $\vartheta^2, \vartheta^3 \in \mathbb{R}$ such that $\vartheta^1 < \vartheta^2 < \vartheta^3$ and $t < \bar{T}(\vartheta^3, \vartheta^2)$. The latter is possible since \bar{T} depends only on the difference $\vartheta_3 - \vartheta_2$, see (4.33). For the fixed t , $k_t \in \mathcal{K}_{\vartheta^1}^* \hookrightarrow \mathcal{K}_{\vartheta^3}^*$, and hence one can write

$$\begin{aligned}
u_t &= \bar{S}_{\vartheta^3\vartheta^*}(t)u_0 \\
&= (u_0 - k_0) + k_t + \int_0^t \bar{S}_{\vartheta^3\vartheta^2}(t-s) D_{\vartheta^2\vartheta^1} k_s ds,
\end{aligned} \tag{4.46}$$

where

$$D_{\vartheta\vartheta''} = \bar{L}_{\vartheta\vartheta''} - L_{\vartheta\vartheta''}^{\Delta}, \quad D_{\vartheta} = \bar{L}_{\vartheta} - L_{\vartheta}^{\Delta},$$

and the latter two operators are as in (4.37) and (3.11) respectively. By Lemma 4.4, for $s \leq t$, $k_s \in \mathcal{K}_{\vartheta^1}^*$. By (2.18), (4.30), and Lemma 4.7 we have that $D_{\vartheta^2\vartheta^1} : \mathcal{K}_{\vartheta^1}^* \rightarrow \mathcal{K}_{\vartheta^2}^+$. Then by Lemma 4.5 the third summand in the second line in (4.46) is in $\mathcal{K}_{\vartheta^3}^+$ which completes the proof since $u_0 - k_0$ is also positive. \square

Proof of Theorem 3.5. According to Definition 3.3 and Remark 3.4 the map $[0, +\infty) \ni t \mapsto k_t \in \mathcal{K}^*$ is the solution in question if: (a) $k_t(\emptyset) = 1$; (b) for each $t > 0$, there exists $\vartheta'' \in \mathbb{R}$ such that $k_t \in \mathcal{K}_{\vartheta''}$ and $\frac{d}{dt}k_t = L_{\vartheta}^{\Delta}k_t$ for each $\vartheta > \vartheta''$.

Let k_0 and $C > 0$ be as in the statement of Theorem 3.5. Set $\vartheta^* = \log C$. Then, for $\vartheta = \vartheta^* + \delta(\vartheta^*)$, see (4.2) and (4.3), k_t as given in (4.45) is a unique solution of (3.11) in \mathcal{K}_ϑ on the time interval $[0, T(\vartheta, \vartheta^*)]$. By (2.18) we have

$$\left(\frac{d}{dt}k_t\right)(\emptyset) = (L^\Delta k_t)(\emptyset) = 0,$$

which yields that $k_t(\emptyset) = k_0(\emptyset) = 1$. By Lemma 4.4 $k_t \in \mathcal{K}_\vartheta^*$, and hence k_t is the solution in question for $t < \tau(\vartheta^*)$. According to Lemma 4.8 k_t lies in $\mathcal{K}_{\vartheta(t)}$ with $\vartheta(t)$ given in (4.40). Fix any $\epsilon \in (0, 1)$ and then set $s_0 = 0$, $s_1 = (1 - \epsilon)\tau(\vartheta^*)$, and $\vartheta_1^* = \vartheta(s_1)$. Thereafter, set $\vartheta^1 = \vartheta_1^* + \delta(\vartheta_1^*)$ and

$$k_{t+s_1} = S_{\vartheta^1 \vartheta_1^*}(t)k_{s_1}, \quad t \in [0, \tau(\vartheta_1^*)].$$

Note that for t such that $t + s_1 < \tau(\vartheta^*)$,

$$k_{t+s_1} = S_{\vartheta^1 \vartheta^*}(t + s_1)k_0,$$

see (4.11). Thus, by Lemmas 4.4 and 4.8 the map $[0, s_1 + \tau(\vartheta_1^*)] \ni t \mapsto k_t \in \mathcal{K}_{\vartheta(t)}$ with

$$k_t = \begin{cases} S_{\vartheta^1 \vartheta^*}(t)k_0 & t \leq s_1; \\ S_{\vartheta^1 \vartheta_1^*}(t - s_1)k_{s_1} & t \in [s_1, s_1 + \tau(\vartheta_1^*)] \end{cases}$$

is the solution in question on the indicated time interval. We continue this procedure by setting $s_n = (1 - \epsilon)\tau(\vartheta_{n-1}^*)$, $n \geq 2$, and then

$$\vartheta_n^* = \vartheta(s_1 + \dots + s_n), \quad \vartheta^n = \vartheta_n^* + \delta(\vartheta_n^*). \quad (4.47)$$

This yields the solution in question on the time interval $[0, s_1 + \dots + s_{n+1}]$ which for $t \in [s_1 + \dots + s_l, s_1 + \dots + s_{l+1}]$, $l = 0, \dots, n$, is given by

$$k_t = S_{\vartheta^l \vartheta_l^*}(t - (s_1 + \dots + s_l))k_{s_l}.$$

Then the global solution in question exists whenever the series

$$\sum_{n \geq 1} s_n = (1 - \epsilon) \sum_{n \geq 1} \tau(\vartheta_n^*)$$

diverges. Assume that this is not the case. Then by (4.40) and (4.47) we get that both (a) and (b) ought to be true, where (a) $\sup_{n \geq 1} \vartheta_n^* =: \bar{\vartheta} < +\infty$ and (b) $\tau(\vartheta_n^*) \rightarrow 0$ as $n \rightarrow +\infty$. However, by (4.2) and (4.3) it follows that (a) implies $\tau(\vartheta_n^*) \geq \tau(\bar{\vartheta}) > 0$, which contradicts (b). \square

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